Pedagogical formal and functionnal systems

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Introduction to formal pedagogy

Published works

📄 L. Colson and D. Michel. *Pedagogical natural deduction systems : the propositional case*. JUCS, 13(10) :1396–1410, 2007.

📄 L. Colson and D. Michel. *Pedagogical Second-Order Propositional Calculi*. JLC, 18(4) :669–695, 2008.

📄 L. Colson and D. Michel. *Pedagogical second-order λ-calculus*. TCS, 410 :4190–4203, 2009.

📄 L. Colson and V. Demange. *Investigations on a Pedagogical Calculus of Constructions*. JUCS, *to appear*, 2013.

Introduction to formal pedagogy
Basics : the pedagogical constraint

Poincaré criterion

«*A definition by postulate has value **only if** there exists an example.*» [Henri Poincaré – Last Thoughts (1913)]

Good postulate

Let x be a natural number such that $x^2 - 1 = 0$ holds.

$\triangleright x := 1$ suits

Bad postulate

Let y be a natural number verifying $y^2 + 1 = 0$.

$\triangleright$ no example

Called "pedagogical" because of usual teaching practice

Outline of the investigation

1. Summary of the previous works
2. First attempts toward a pedagogical CC
3. A formal definition and some examples
4. Toward a Pedagogical Calculus of Constructions

Summary of the previous works
The formal pedagogical constraint

Poincaré criterion (informally)

> «*A definition by postulate has value **only if** there exists an example.*» [Henri Poincaré – Last Thoughts (1913)]

Poincaré criterion (formally)

If $\top$ is a type and o a term of type $\top$:

- $\Gamma := \{ f : (\alpha \to \alpha) \to \alpha, g : \top \to \beta \}$ $\qquad \triangleright$ defines α and β
- $\sigma := [\alpha \mapsto \top, \beta \mapsto \top]$ exemplifies Γ :

$$\vdash \lambda h^{\top \to \top}.o : (\top \to \top) \to \top$$
$$\vdash \lambda x^{\top}.x \quad : \top \to \top$$

$\vdash \sigma \cdot \Gamma$ abbreviates that σ exemplifies Γ

Summary of the previous works
Simply typed λ-calculus

Morphology $\alpha \mid A \to B$

Syntax $x \mid \lambda x^A.u \mid u\,v$

$$\frac{x : F \in \Gamma}{\Gamma \vdash x : F} \ (\text{var})$$

$$\frac{\Gamma, x : A \vdash u : B}{\Gamma \vdash \lambda x^A.u : A \to B} \ (\text{abs}) \qquad \frac{\Gamma \vdash u : A \to B \quad \Gamma \vdash v : A}{\Gamma \vdash u\,v : B} \ (\text{app})$$

Summary of the previous works

Pedagogical simply typed λ-calculus [Colson and Michel (2007)]

Morphology $\alpha \mid A \to B$

  Syntax $x \mid \lambda x^A.u \mid u\ v$

$$\frac{x : F \in \Gamma \quad \vdash \sigma \cdot \Gamma}{\Gamma \vdash x : F}\ \text{(var)}$$

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Summary of the previous works

Pedagogical simply typed λ-calculus [Colson and Michel (2007)]

Morphology $\alpha \mid A \to B$

Syntax $x \mid \lambda x^A.u \mid u\ v$

No starting rule

$$\frac{x : F \in \Gamma \quad \vdash \sigma \cdot \Gamma}{\Gamma \vdash x : F}\ (\text{var})$$

$$\frac{\Gamma, x : A \vdash u : B}{\Gamma \vdash \lambda x^A.u : A \to B}\ (\text{abs}) \qquad \frac{\Gamma \vdash u : A \to B \quad \Gamma \vdash v : A}{\Gamma \vdash u\ v : B}\ (\text{app})$$

Summary of the previous works

Pedagogical simply typed λ-calculus [Colson and Michel (2007)]

Morphology $\alpha \mid A \to B \mid \top$

Syntax $x \mid \lambda x^A.u \mid u\ v \mid o$

$$\frac{\vdash \sigma \cdot \Gamma}{\Gamma \vdash o : \top}\ \text{(ax)} \qquad \frac{x : F \in \Gamma \quad \vdash \sigma \cdot \Gamma}{\Gamma \vdash x : F}\ \text{(var)}$$

$$\frac{\Gamma, x : A \vdash u : B}{\Gamma \vdash \lambda x^A.u : A \to B}\ \text{(abs)} \qquad \frac{\Gamma \vdash u : A \to B \quad \Gamma \vdash v : A}{\Gamma \vdash u\ v : B}\ \text{(app)}$$

Summary of the previous works
Pedagogical simply typed λ-calculus [Colson and Michel (2007)]

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$$\frac{\Gamma, x : A \vdash u : B}{\Gamma \vdash \lambda x^A.u : A \to B}\ (\text{abs}) \qquad \frac{\Gamma \vdash u : A \to B \quad \Gamma \vdash v : A}{\Gamma \vdash u\ v : B}\ (\text{app})$$

Results

- all formulas exemplified by $\top$
- initial and pedagogical systems (syntactically) equivalents

Summary of the previous works
System F [Girard (1972), Reynolds (1974)]

Morphology $\alpha \mid A \to B \mid \forall \alpha.A$

Syntax $x \mid \lambda x^A.u \mid u\ v \mid \Lambda \alpha.u$

$$\frac{x : F \in \Gamma}{\Gamma \overset{f}{\vdash} x : F} \text{ (var)}$$

$$\frac{\Gamma, x : A \overset{f}{\vdash} u : B}{\Gamma \overset{f}{\vdash} \lambda x^A.u : A \to B} \text{ (abs)} \qquad \frac{\Gamma \overset{f}{\vdash} u : A \to B \quad \Gamma \overset{f}{\vdash} v : A}{\Gamma \overset{f}{\vdash} u\ v : B} \text{ (app)}$$

$$\frac{\Gamma \overset{f}{\vdash} u : B \quad \alpha \notin \mathcal{V}(\Gamma)}{\Gamma \overset{f}{\vdash} \Lambda \alpha.u : \forall \alpha.B} \text{ (Abs)} \qquad \frac{\Gamma \overset{f}{\vdash} u : \forall \alpha.B}{\Gamma \overset{f}{\vdash} u\ V : B[\alpha \leftarrow V]} \text{ (App)}$$

Summary of the previous works
Weakly pedagogical System F [Colson and Michel (2008)]

Morphology $\alpha \mid A \to B \mid \forall \alpha.A \mid \top$
     Syntax $x \mid \lambda x^A.u \mid u\ v \mid \Lambda \alpha.u \mid o$

$$\frac{\vdash^{\mathsf{fw}} \sigma \cdot \Gamma}{\Gamma \vdash^{\mathsf{fw}} o : \top} \ (\mathsf{ax}) \qquad \frac{x : F \in \Gamma \quad \vdash^{\mathsf{fw}} \sigma \cdot \Gamma}{\Gamma \vdash^{\mathsf{fw}} x : F} \ (\mathsf{var})$$

$$\frac{\Gamma, x : A \vdash^{\mathsf{fw}} u : B}{\Gamma \vdash^{\mathsf{fw}} \lambda x^A.u : A \to B} \ (\mathsf{abs}) \qquad \frac{\Gamma \vdash^{\mathsf{fw}} u : A \to B \quad \Gamma \vdash^{\mathsf{fw}} v : A}{\Gamma \vdash^{\mathsf{fw}} u\ v : B} \ (\mathsf{app})$$

$$\frac{\Gamma \vdash^{\mathsf{fw}} u : B \quad \alpha \notin \mathcal{V}(\Gamma)}{\Gamma \vdash^{\mathsf{fw}} \Lambda \alpha.u : \forall \alpha.B} \ (\mathsf{Abs}) \qquad \frac{\Gamma \vdash^{\mathsf{fw}} u : \forall \alpha.B}{\Gamma \vdash^{\mathsf{fw}} u\ V : B[\alpha \leftarrow V]} \ (\mathsf{App})$$

Morphology $\quad \alpha \mid A \to B \mid \forall \alpha.A \mid \top$

Syntax $\quad x \mid \lambda x^A.u \mid u\ v \mid \Lambda \alpha.u \mid o$

$$\frac{\vdash^{\mathsf{fw}} \sigma \cdot \Gamma}{\Gamma \vdash^{\mathsf{fw}} o : \top} \ (\mathsf{ax}) \qquad \frac{x : F \in \Gamma \quad \vdash^{\mathsf{fw}} \sigma \cdot \Gamma}{\Gamma \vdash^{\mathsf{fw}} x : F} \ (\mathsf{var})$$

$$\frac{\Gamma, x : A \vdash^{\mathsf{fw}} u : B}{\Gamma \vdash^{\mathsf{fw}} \lambda x^A.u : A \to B} \ (\mathsf{abs}) \qquad \frac{\Gamma \vdash^{\mathsf{fw}} u : A \to B \quad \Gamma \vdash^{\mathsf{fw}} v : A}{\Gamma \vdash^{\mathsf{fw}} u\ v : B} \ (\mathsf{app})$$

$$\frac{\Gamma \vdash^{\mathsf{fw}} u : B \quad \alpha \notin \mathcal{V}(\Gamma)}{\Gamma \vdash^{\mathsf{fw}} \Lambda \alpha.u : \forall \alpha.B} \ (\mathsf{Abs}) \qquad \frac{\Gamma \vdash^{\mathsf{fw}} u : \forall \alpha.B}{\Gamma \vdash^{\mathsf{fw}} u\ V : B[\alpha \leftarrow V]} \ (\mathsf{App})$$

No subject reduction property :

$\vdash^{\mathsf{fw}} (\Lambda \alpha.\lambda x^\alpha.x) \perp : \perp \to \perp$ but

$\quad\quad \not\vdash^{\mathsf{fw}} \lambda x^\perp.x : \perp \to \perp \qquad\qquad \triangleright \perp := \forall \alpha.\alpha$

Summary of the previous works
Pedagogical System F [Colson and Michel (2009)]

Pre-morphology $\alpha \mid A \to B \mid \forall \alpha.A \mid \top$
Syntax $x \mid \lambda x^A.u \mid u\ v \mid \Lambda \alpha.u \mid o$

$$\frac{\vdash^{\mathsf{fp}} \sigma \cdot \Gamma}{\Gamma \vdash^{\mathsf{fp}} o : \top} \text{ (ax)} \qquad \frac{x : F \in \Gamma \quad \vdash^{\mathsf{fp}} \sigma \cdot \Gamma}{\Gamma \vdash^{\mathsf{fp}} x : F} \text{ (var)}$$

$$\frac{\Gamma, x : A \vdash^{\mathsf{fp}} u : B}{\Gamma \vdash^{\mathsf{fp}} \lambda x^A.u : A \to B} \text{ (abs)} \qquad \frac{\Gamma \vdash^{\mathsf{fp}} u : A \to B \quad \Gamma \vdash^{\mathsf{fp}} v : A}{\Gamma \vdash^{\mathsf{fp}} u\ v : B} \text{ (app)}$$

$$\frac{\Gamma \vdash^{\mathsf{fp}} u : B \quad \alpha \notin \mathcal{V}(\Gamma)}{\Gamma \vdash^{\mathsf{fp}} \Lambda \alpha.u : \forall \alpha.B} \text{ (Abs)} \qquad \frac{\Gamma \vdash^{\mathsf{fp}} u : \forall \alpha.B \quad \vdash^{\mathsf{fp}} \sigma \cdot V}{\Gamma \vdash^{\mathsf{fp}} u\ V : B[\alpha \leftarrow V]} \text{ (App)}$$

Summary of the previous works
Pedagogical System F [Colson and Michel (2009)]

Main results

- every well-formed sub-type can be exemplified :
    - Poincaré criterion always satisfied :
      $$\Gamma \vdash^{\mathbf{fp}} u : A \quad \Rightarrow \quad \vdash^{\mathbf{fp}} \sigma \cdot \Gamma$$
    - usefulness of functions :
      $$\Gamma \vdash^{\mathbf{fp}} f : A \to B \text{ with } A \text{ closed} \quad \Rightarrow \quad A \text{ is inhabited}$$
    - negationless second-order propositional calculus
- embeddings from System F to its pedagogical version :
    - at logical side :
      $$\exists t \quad \Gamma \vdash^{\mathbf{f}} t : F \quad \Leftrightarrow \quad \exists t' \quad \Gamma^\gamma \vdash^{\mathbf{fp}} t' : F^\gamma$$
      where F^γ is F with occurrences of variables α replaced by $\alpha \vee \gamma$ (γ fresh)
    - at computational side :
      $$\Gamma \vdash^{\mathbf{f}} t : A \quad \Rightarrow \quad \Gamma^\delta \vdash^{\mathbf{fp}} \bar{t} : A^\delta$$
      where A^δ is a *double-negation* version of A ($\bot$ replaced by δ)
      and $\bar{t}$ is a CPS version of t (δ fresh)

Summary of the previous works
Pedagogical System F [Colson and Michel (2009)]

Main results

logic	computation
formula	datatype
proof	program
simplification	computation
proof checking	type checking
exemplifiability	utility

First attempts toward a pedagogical CC
Calculus of Constructions [Coquand and Huet (1985)]

Motivations

The Calculus of Constructions :

- uniform presentation of previous systems
- interdependence between morphology and syntax
    - ▷ reduction of constraints number
- first step towards pedagogical pure type systems (PTS)
    - ▷ especially Type : Type [Martin-Löf (1971)]

First attempts toward a pedagogical CC
Calculus of Constructions [Coquand and Huet (1985)]

$$\frac{}{\emptyset \; \mathsf{wf}^{\mathsf{c}}} \; \text{(c-env}_1\text{)} \qquad \frac{\Gamma \vDash^{\mathsf{c}} A : \kappa \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : A \, \mathsf{wf}^{\mathsf{c}}} \; \text{(c-env}_2\text{)}$$

$$\frac{\Gamma \, \mathsf{wf}^{\mathsf{c}}}{\Gamma \vDash^{\mathsf{c}} \mathsf{Prop} : \mathsf{Type}} \; \text{(c-ax)} \qquad \frac{\Gamma, x : A, \Gamma' \, \mathsf{wf}^{\mathsf{c}}}{\Gamma, x : A, \Gamma' \vDash^{\mathsf{c}} x : A} \; \text{(c-var)}$$

$$\frac{\Gamma, x : A \vDash^{\mathsf{c}} u : B : \kappa}{\Gamma \vDash^{\mathsf{c}} \lambda x^A.u : \forall x^A.B} \; \text{(c-abs)} \qquad \frac{\Gamma \vDash^{\mathsf{c}} u : \forall x^A.B \quad \Gamma \vDash^{\mathsf{c}} v : A}{\Gamma \vDash^{\mathsf{c}} u \; v : B[x \leftarrow v]} \; \text{(c-app)}$$

$$\frac{\Gamma, x : A \vDash^{\mathsf{c}} B : \kappa}{\Gamma \vDash^{\mathsf{c}} \forall x^A.B : \kappa} \; \text{(c-prod)} \qquad \frac{\Gamma \vDash^{\mathsf{c}} u : A \quad A \simeq A' \quad \Gamma \vDash^{\mathsf{c}} A' : \kappa}{\Gamma \vDash^{\mathsf{c}} u : A'} \; \text{(c-conv)}$$

κ denotes Prop or Type

First attempts toward a pedagogical CC
Naive transposition

Formal Poincaré criterion

$\Gamma \equiv x_1 : A_1, \ldots, x_n : A_n$ can be exemplified by terms $t_1, \ldots, t_n$ if

$$\vdash t_1 : A_1$$
$$\vdash t_2 : A_2[x_1 \leftarrow t_1]$$
$$\vdots$$
$$\vdash t_n : A_n[x_1, \ldots, x_{n-1} \leftarrow t_1, \ldots, t_{n-1}]$$

Abbreviation : $\vdash \sigma \cdot \Gamma$ with $\sigma := [x_1 \mapsto t_1, \ldots, x_n \mapsto t_n]$

First idea

Substitutes the well-formedness of an environment by its
exemplifiability $\triangleright \Gamma$ wf   replaced by $\vdash \sigma \cdot \Gamma$

First attempts toward a pedagogical CC
Naive transposition

$$\frac{\vdash^{\mathsf{n}} \sigma \cdot \Gamma}{\Gamma \vdash^{\mathsf{n}} o : \top : \mathsf{Prop} : \mathsf{Type}} \text{ (n-ax)} \qquad \frac{\vdash^{\mathsf{n}} \sigma \cdot (\Gamma, x : A, \Gamma')}{\Gamma, x : A, \Gamma' \vdash^{\mathsf{n}} x : A} \text{ (n-var)}$$

But

- "exemplifiable" does not imply "well-formed" :
  - $x_1 : \mathsf{Type}$ $\qquad\qquad$ $\triangleright \sigma := [x_1 \mapsto \mathsf{Prop}]$
  - $x_1 : \mathsf{Prop}, x_2 : (\lambda H^{\top \to x_1}.\top) \; (\lambda y^{\top}.y)$ $\qquad \triangleright \sigma := [x_1 \mapsto \top; x_2 \mapsto o]$
  - etc.

- no subject reduction $\qquad\qquad\qquad$ $\triangleright \bot \to \bot$ inhabited

First attempts toward a pedagogical CC
Calculus of Constructions [Coquand and Huet (1985)]

$$\frac{}{\emptyset \; \mathsf{wf}^{\mathsf{c}}} \; (\text{c-env}_1) \qquad \frac{\Gamma \vDash^{\mathsf{c}} A : \kappa \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : A \, \mathsf{wf}^{\mathsf{c}}} \; (\text{c-env}_2)$$

$$\frac{\Gamma \, \mathsf{wf}^{\mathsf{c}}}{\Gamma \vDash^{\mathsf{c}} \mathsf{Prop} : \mathsf{Type}} \; (\text{c-ax}) \qquad \frac{\Gamma, x : A, \Gamma' \, \mathsf{wf}^{\mathsf{c}}}{\Gamma, x : A, \Gamma' \vDash^{\mathsf{c}} x : A} \; (\text{c-var})$$

$$\frac{\Gamma, x : A \vDash^{\mathsf{c}} u : B : \kappa}{\Gamma \vDash^{\mathsf{c}} \lambda x^A.u : \forall x^A.B} \; (\text{c-abs}) \qquad \frac{\Gamma \vDash^{\mathsf{c}} u : \forall x^A.B \quad \Gamma \vDash^{\mathsf{c}} v : A}{\Gamma \vDash^{\mathsf{c}} u \; v : B[x \leftarrow v]} \; (\text{c-app})$$

$$\frac{\Gamma, x : A \vDash^{\mathsf{c}} B : \kappa}{\Gamma \vDash^{\mathsf{c}} \forall x^A.B : \kappa} \; (\text{c-prod}) \qquad \frac{\Gamma \vDash^{\mathsf{c}} u : A \quad A \simeq A' \quad \Gamma \vDash^{\mathsf{c}} A' : \kappa}{\Gamma \vDash^{\mathsf{c}} u : A'} \; (\text{c-conv})$$

κ denotes Prop or Type

First attempts toward a pedagogical CC
Calculus of Constructions [Coquand and Huet (1985)]

$$\frac{}{\emptyset \; \mathsf{wf}^c} \; (\text{c-env}_1)$$

$$\frac{\Gamma \vdash^c A : \kappa \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : A \; \mathsf{wf}^c} \; (\text{c-env}_2)$$

$$\frac{\Gamma \; \mathsf{wf}^c}{\Gamma \vdash^c \mathsf{Prop} : \mathsf{Type}} \; (\text{c-ax})$$

$$\frac{\Gamma, x : A, \Gamma' \; \mathsf{wf}^c}{\Gamma, x : A, \Gamma' \vdash^c x : A} \; (\text{c-var})$$

$$\frac{\Gamma, x : A \vdash^c u : B : \kappa}{\Gamma \vdash^c \lambda x^A.u : \forall x^A.B} \; (\text{c-abs})$$

$$\frac{\Gamma \vdash^c u : \forall x^A.B \quad \Gamma \vdash^c v : A}{\Gamma \vdash^c u \; v : B[x \leftarrow v]} \; (\text{c-app})$$

$$\frac{\Gamma, x : A \vdash^c B : \kappa}{\Gamma \vdash^c \forall x^A.B : \kappa} \; (\text{c-prod})$$

$$\frac{\Gamma \vdash^c u : A \quad A \simeq A' \quad \Gamma \vdash^c A' : \kappa}{\Gamma \vdash^c u : A'} \; (\text{c-conv})$$

(c-prod) responsible of vacuity κ denotes Prop or Type

First attempts toward a pedagogical CC
Poincarean Calculus of Constructions – CCr

Strategy change

- keep Γ wf judgements
- avoid empty types as soon as possible

Radical idea
Constrain (only) the type formation rule :

$$\frac{\Gamma, x : A \vdash^{\mathsf{c}} \quad B : \kappa}{\Gamma \vdash^{\mathsf{c}} \forall x^A.B : \kappa} \text{ (c-prod)}$$

First attempts toward a pedagogical CC
Poincarean Calculus of Constructions – CCr

Strategy change

- keep Γ wf judgements
- avoid empty types as soon as possible

Radical idea
Constrain (only) the type formation rule :

$$\frac{\Gamma, x : A \vdash^r u : B : \kappa}{\Gamma \vdash^r \forall x^A. B : \kappa} \text{ (r-prod)}$$

First attempts toward a pedagogical CC
Poincarean Calculus of Constructions – CCr

Theorems

- ▶ respectful of the Poincaré criterion
  - ▷ relies on strong normalization of CC
- ▶ subject reduction property holds
  - ▷ Coq proof adapted from [Barras (1996)]
- ▶ usefulness of functions
  - ▷ $\vdash^{\mathsf{r}} f : \forall x^A.B \quad \Rightarrow \quad \exists u \quad \vdash^{\mathsf{r}} u : A$
- ▶ terms of Gödel system T can be interpreted in CC_r
  - ▷ usual way : recursion from iteration (and trick for cartesian products)

But

- ▶ does not contain natively simply typed λ-calculus
  - ▷ $\nexists u \quad A\ B\ C : \mathsf{Prop} \vdash^{\mathsf{r}} u : (A \to B) \to (B \to C) \to (A \to C)$
- ▶ does not prove symmetry of Leibniz equality
  - ▷ $\nexists u \quad A : \mathsf{Prop}, x\ y\ z : A \vdash^{\mathsf{r}} u : x =_A y \to y =_A z \to x =_A z$

First attempts toward a pedagogical CC
Poincarean Calculus of Constructions – CCr

Observation CCr seems too much constrained

Goal increase expressive power

Idea exemplifiable types should be usable
  ▷ similar to pedagogical system F

Definition : converse of the Poincaré criterion

$CC_\star$, sub-system of CC, meets the *converse of the Poincaré criterion* if :

$$\vdash^\star \sigma \cdot \Gamma \qquad \Rightarrow \qquad \Gamma \, \text{wf}^\star$$

First attempts toward a pedagogical CC
Poincarean Calculus of Constructions – CCr

Observation	CCr seems too much constrained
Goal	increase expressive power
Idea	exemplifiable types should be usable
	▷ similar to pedagogical system F

Beware
Exemplifiable types need not be well-formed

Definition : converse of the Poincaré criterion
$CC_\star$, sub-system of CC, meets the *converse of the Poincaré criterion* if :

$$\vdash^\star \sigma \cdot \Gamma \text{ and } \Gamma \text{ wf}^\mathsf{c} \quad \Rightarrow \quad \Gamma \text{ wf}^\star$$

A formal definition and some examples
Pedagogical sub-system of the Calculus of Constructions

Definition : $CC_\star$ pedagogical sub-system of CC

- $CC_\star$ sub-system of CC

  $\triangleright \Gamma \vdash^\star u : A \quad \Rightarrow \quad \Gamma \vdash^c u : A$

- Subject reduction property holds for $CC_\star$

  $\triangleright \Gamma \vdash^\star u : A$ and $u \rightsquigarrow u' \quad \Rightarrow \quad \Gamma \vdash^\star u' : A$

- $CC_\star$ meets Poincaré criterion and its converse

  $\triangleright \Gamma \, wf^\star \quad \Leftrightarrow \quad \vdash^\star \sigma \cdot \Gamma$ and $\Gamma \, wf^c$

A formal definition and some examples
Calculus of Constructions [Coquand and Huet (1985)]

$$\frac{}{\emptyset \, \mathsf{wf}^{\mathsf{c}}} \; (\text{c-env}_1) \qquad \frac{\Gamma \vdash^{\mathsf{c}} A : \kappa \quad x \notin \mathsf{dom}(\Gamma)}{\Gamma, x : A \, \mathsf{wf}^{\mathsf{c}}} \; (\text{c-env}_2)$$

$$\frac{\Gamma \, \mathsf{wf}^{\mathsf{c}}}{\Gamma \vdash^{\mathsf{c}} \mathsf{Prop} : \mathsf{Type}} \; (\text{c-ax}) \qquad \frac{\Gamma, x : A, \Gamma' \, \mathsf{wf}^{\mathsf{c}}}{\Gamma, x : A, \Gamma' \vdash^{\mathsf{c}} x : A} \; (\text{c-var})$$

$$\frac{\Gamma, x : A \vdash^{\mathsf{c}} u : B : \kappa}{\Gamma \vdash^{\mathsf{c}} \lambda x^A.u : \forall x^A.B} \; (\text{c-abs}) \qquad \frac{\Gamma \vdash^{\mathsf{c}} u : \forall x^A.B \quad \Gamma \vdash^{\mathsf{c}} v : A}{\Gamma \vdash^{\mathsf{c}} u\,v : B[x \leftarrow v]} \; (\text{c-app})$$

$$\frac{\Gamma, x : A \vdash^{\mathsf{c}} B : \kappa}{\Gamma \vdash^{\mathsf{c}} \forall x^A.B : \kappa} \; (\text{c-prod}) \qquad \frac{\Gamma \vdash^{\mathsf{c}} u : A \quad A \simeq A' \quad \Gamma \vdash^{\mathsf{c}} A' : \kappa}{\Gamma \vdash^{\mathsf{c}} u : A'} \; (\text{c-conv})$$

κ denotes Prop or Type

A formal definition and some examples
Second order λ-calculus $- \lambda^2$

$$\frac{}{\emptyset \, \text{wf}^2} \; (\; \text{env}_1)$$

$$\frac{\Gamma \vdash^2 A : \kappa \quad x \notin \text{dom}(\Gamma)}{\Gamma, x : A \, \text{wf}^2} \; (\; \text{env}_2)$$

$$\frac{\Gamma \, \text{wf}^2}{\Gamma \vdash^2 \text{Prop} : \text{Type}} \; (\; \text{ax})$$

$$\frac{\Gamma, x : A, \Gamma' \, \text{wf}^2}{\Gamma, x : A, \Gamma' \vdash^2 x : A} \; (\; \text{var})$$

$$\frac{\Gamma, x : A \vdash^2 u : B : \text{Prop}}{\Gamma \vdash^2 \lambda x^A.u : \forall x^A.B} \; (\; \text{abs})$$

$$\frac{\Gamma \vdash^2 u : \forall x^A.B \quad \Gamma \vdash^2 v : A}{\Gamma \vdash^2 u \, v : B[x \leftarrow v]} \; (\; \text{app})$$

$$\frac{\Gamma, x : A \vdash^2 B : \text{Prop}}{\Gamma \vdash^2 \forall x^A.B : \text{Prop}} \; (\; \text{prod})$$

Goal

Obtain a pedagogical version of λ^2

$\triangleright$ in the sense of the previous definition

First idea

Keep examples explicit : $\Gamma \vdash_\sigma u : A$ and $\Gamma \, \text{wf}_\sigma$ where

- $\Gamma \equiv x_1 : A_1, \ldots, x_n : A_n$
- $\sigma \equiv [x_1 \mapsto t_1, \ldots, x_n \mapsto t_n]$ or in short $\sigma \equiv [t_1, \ldots, t_n]$

A formal definition and some examples
Second order λ-calculus $-\lambda^2$

$$\frac{}{\emptyset \, \mathsf{wf}^2} \; (\text{ env}_1) \qquad \frac{\Gamma \vdash^2 \; A : \kappa \qquad\qquad x \notin \text{dom}(\Gamma)}{\Gamma, x : A \, \mathsf{wf}^2} \; (\text{ env}_2)$$

$$\frac{\Gamma \, \mathsf{wf}^2}{\Gamma \vdash^2 \qquad \mathsf{Prop} : \mathsf{Type}} \; (\text{ ax}) \qquad \frac{\Gamma, x : A, \Gamma' \, \mathsf{wf}^2}{\Gamma, x : A, \Gamma' \vdash^2 \; x : A} \; (\text{ var})$$

$$\frac{\Gamma, x : A \vdash^2 \quad u : B : \mathsf{Prop}}{\Gamma \vdash^2 \; \lambda x^A.u : \forall x^A.B} \; (\text{ abs}) \qquad \frac{\Gamma \vdash^2 \; u : \forall x^A.B \quad \Gamma \vdash^2 \; v : A}{\Gamma \vdash^2 \; u \; v : B[x \leftarrow v]} \; (\text{ app})$$

$$\frac{\Gamma, x : A \vdash^2 \quad B : \mathsf{Prop}}{\Gamma \vdash^2 \; \forall x^A.B : \mathsf{Prop}} \; (\text{ prod})$$

A formal definition and some examples

With explicit total exemplifications – λ_e^2

$$\frac{}{\emptyset \, \mathsf{wf}_{\emptyset}^{2e}} \; (\text{e-env}_1)$$

$$\frac{\Gamma \vDash_{\sigma}^{2e} A : \kappa \qquad\qquad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : A \, \mathsf{wf}_{\sigma :: a}^{2e}} \; (\text{e-env}_2)$$

$$\frac{\Gamma \, \mathsf{wf}_{\sigma}^{2e}}{\Gamma \vDash_{\sigma}^{2e} \quad \mathsf{Prop} : \mathsf{Type}} \; (\text{e-ax})$$

$$\frac{\Gamma, x : A, \Gamma' \, \mathsf{wf}_{\sigma}^{2e}}{\Gamma, x : A, \Gamma' \vDash_{\sigma}^{2e} x : A} \; (\text{e-var})$$

$$\frac{\Gamma, x : A \vDash_{\sigma :: a}^{2e} u : B : \mathsf{Prop}}{\Gamma \vDash_{\sigma}^{2e} \lambda x^A.u : \forall x^A.B} \; (\text{e-abs})$$

$$\frac{\Gamma \vDash_{\sigma}^{2e} u : \forall x^A.B \quad \Gamma \vDash_{\sigma}^{2e} v : A}{\Gamma \vDash_{\sigma}^{2e} u \; v : B[x \leftarrow v]} \; (\text{e-app})$$

$$\frac{\Gamma, x : A \vDash_{\sigma :: a}^{2e} B : \mathsf{Prop}}{\Gamma \vDash_{\sigma}^{2e} \forall x^A.B : \mathsf{Prop}} \; (\text{e-prod})$$

$$\frac{}{\emptyset \,\text{wf}_{\emptyset}^{2e}}\ (\text{e-env}_1)$$

$$\frac{\Gamma \mathrel{|\!\!\frac{2e}{\sigma}} A : \kappa \quad \mathrel{|\!\!\frac{2e}{\emptyset}} a : \sigma(A) \quad x \notin \text{dom}(\Gamma)}{\Gamma, x : A \,\text{wf}_{\sigma::\,a}^{2e}}\ (\text{e-env}_2)$$

$$\frac{\Gamma \,\text{wf}_{\sigma}^{2e}}{\Gamma \mathrel{|\!\!\frac{2e}{\sigma}} o : \top : \text{Prop} : \text{Type}}\ (\text{e-ax})$$

$$\frac{\Gamma, x : A, \Gamma' \,\text{wf}_{\sigma}^{2e}}{\Gamma, x : A, \Gamma' \mathrel{|\!\!\frac{2e}{\sigma}} x : A}\ (\text{e-var})$$

$$\frac{\Gamma, x : A \mathrel{|\!\!\frac{2e}{\sigma::\,a}} u : B : \text{Prop}}{\Gamma \mathrel{|\!\!\frac{2e}{\sigma}} \lambda x^A.u : \forall x^A.B}\ (\text{e-abs})$$

$$\frac{\Gamma \mathrel{|\!\!\frac{2e}{\sigma}} u : \forall x^A.B \quad \Gamma \mathrel{|\!\!\frac{2e}{\sigma}} v : A}{\Gamma \mathrel{|\!\!\frac{2e}{\sigma}} u\ v : B[x \leftarrow v]}\ (\text{e-app})$$

$$\frac{\Gamma, x : A \mathrel{|\!\!\frac{2e}{\sigma::\,a}} B : \text{Prop} \quad \mathrel{|\!\!\frac{2e}{\emptyset}} t : \sigma(\forall x^A.B)}{\Gamma \mathrel{|\!\!\frac{2e}{\sigma}} \forall x^A.B : \text{Prop}}\ (\text{e-prod})$$

A formal definition and some examples
With explicit total exemplifications – λ_e^2

Theorems

- λ_e^2 (*almost*) a pedagogical sub-system of CC
- examples contained as sub-derivations
  $$\triangleright \Gamma \vdash_\sigma^{2e} A : \kappa \quad \Rightarrow \quad \exists a \quad \vdash_\emptyset^{2e} a : \sigma(A) \text{ sub-derivation}$$
- exchange of exemplifications :
  $$\triangleright \Gamma \vdash_\sigma^{2e} w : C \text{ and } \Gamma \operatorname{wf}_\rho^{2e} \quad \Rightarrow \quad \Gamma \vdash_\rho^{2e} w : C$$

But
λ_e^2 not exactly sub-system of CC because :

- exemplifications are explicit
- addition of two symbols o and $\top$

A formal definition and some examples
With explicit total exemplifications – λ_e^2

Theorems

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But

λ_e^2 not exactly sub-system of CC because :

- ▶ exemplifications are explicit
- ▶ addition of two symbols o and $\top$

Second idea

Relax constraints on exemplifications by making them implicit

A formal definition and some examples

With explicit total exemplifications – λ_e^2

$$\frac{}{\emptyset \, \mathsf{wf}_\emptyset^{\mathsf{2e}}} \; \text{(e-env}_1\text{)}$$

$$\frac{\Gamma \vdash_\sigma^{\mathsf{2e}} A : \kappa \quad \vdash_\emptyset^{\mathsf{2e}} a : \sigma(A) \quad x \notin \mathsf{dom}(\Gamma)}{\Gamma, x : A \, \mathsf{wf}_{\sigma :: \, a}^{\mathsf{2e}}} \; \text{(e-env}_2\text{)}$$

$$\frac{\Gamma \, \mathsf{wf}_\sigma^{\mathsf{2e}}}{\Gamma \vdash_\sigma^{\mathsf{2e}} o : \top : \mathsf{Prop} : \mathsf{Type}} \; \text{(e-ax)}$$

$$\frac{\Gamma, x : A, \Gamma' \, \mathsf{wf}_\sigma^{\mathsf{2e}}}{\Gamma, x : A, \Gamma' \vdash_\sigma^{\mathsf{2e}} x : A} \; \text{(e-var)}$$

$$\frac{\Gamma, x : A \vdash_{\sigma :: \, a}^{\mathsf{2e}} u : B : \mathsf{Prop}}{\Gamma \vdash_\sigma^{\mathsf{2e}} \lambda x^A.u : \forall x^A.B} \; \text{(e-abs)}$$

$$\frac{\Gamma \vdash_\sigma^{\mathsf{2e}} u : \forall x^A.B \quad \Gamma \vdash_\sigma^{\mathsf{2e}} v : A}{\Gamma \vdash_\sigma^{\mathsf{2e}} u \, v : B[x \leftarrow v]} \; \text{(e-app)}$$

$$\frac{\Gamma, x : A \vdash_{\sigma :: \, a}^{\mathsf{2e}} B : \mathsf{Prop} \quad \vdash_\emptyset^{\mathsf{2e}} t : \sigma(\forall x^A.B)}{\Gamma \vdash_\sigma^{\mathsf{2e}} \forall x^A.B : \mathsf{Prop}} \; \text{(e-prod)}$$

$$\frac{}{\emptyset \, \mathsf{wf}^{2t}} \text{ (t-env}_1\text{)}$$

$$\frac{\Gamma \vDash^{2t} A : \kappa \qquad\qquad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : A \, \mathsf{wf}^{2t}} \text{ (t-env}_2\text{)}$$

$$\frac{\Gamma \, \mathsf{wf}^{2t}}{\Gamma \vDash^{2t} o : \top : \mathsf{Prop} : \mathsf{Type}} \text{ (t-ax)}$$

$$\frac{\Gamma, x : A, \Gamma' \, \mathsf{wf}^{2t}}{\Gamma, x : A, \Gamma' \vDash^{2t} x : A} \text{ (t-var)}$$

$$\frac{\Gamma, x : A \vDash^{2t} \quad u : B : \mathsf{Prop}}{\Gamma \vDash^{2t} \lambda x^A.u : \forall x^A.B} \text{ (t-abs)}$$

$$\frac{\Gamma \vDash^{2t} u : \forall x^A.B \quad \Gamma \vDash^{2t} v : A}{\Gamma \vDash^{2t} u \; v : B[x \leftarrow v]} \text{ (t-app)}$$

$$\frac{\Gamma, x : A \vDash^{2t} \quad B : \mathsf{Prop} \qquad \sigma \, \mathsf{mot}_\Gamma \, \forall x^A.B}{\Gamma \vDash^{2t} \forall x^A.B : \mathsf{Prop}} \text{ (t-prod)}$$

where $\sigma \, \mathsf{mot}_\Gamma \, C$ abbreviates :

(a) σ exemplifies Γ, i.e. $\vDash^{2t} \sigma \cdot \Gamma$

(b) and there is term t such that $\vDash^{2t} t : \sigma(C)$

A formal definition and some examples
With implicit total exemplifications – λ_t^2

Theorems
λ_t^2 equivalent to λ_e^2 : $\Gamma \vdash^{2t} t : A \quad \Leftrightarrow \quad \exists \sigma \quad \Gamma \vdash_\sigma^{2e} t : A$

- λ_t^2 (*almost*) a pedagogical sub-system of CC

But
λ_t^2 not exactly sub-system of CC because :

- addition of two symbols o and $\top$

A formal definition and some examples
With implicit total exemplifications – λ_t^2

Theorems
λ_t^2 equivalent to λ_e^2 : $\Gamma \vdash^{2t} t : A \quad \Leftrightarrow \quad \exists \sigma \quad \Gamma \vdash_\sigma^{2e} t : A$

- ▶ λ_t^2 (*almost*) a pedagogical sub-system of CC

But
λ_t^2 not exactly sub-system of CC because :

- ▶ addition of two symbols o and $\top$

Last idea
Have partial exemplifications to restore the CC_r's behaviour :
$\Gamma \vdash^r$ Id : True : Prop

$\qquad \triangleright$ with Id $:= \lambda A^{\mathsf{Prop}}.\lambda x^A.x$ and True $:= \forall A^{\mathsf{Prop}}.A \to A$

A formal definition and some examples
With implicit total exemplifications $- \lambda_t^2$

$$\frac{}{\emptyset \, \mathsf{wf}^{2t}} \; \text{(t-env}_1)$$

$$\frac{\Gamma \models^{2t} A : \kappa \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : A \, \mathsf{wf}^{2t}} \; \text{(t-env}_2)$$

$$\frac{\Gamma \, \mathsf{wf}^{2t}}{\Gamma \models^{2t} o : \top : \mathsf{Prop} : \mathsf{Type}} \; \text{(t-ax)}$$

$$\frac{\Gamma, x : A, \Gamma' \, \mathsf{wf}^{2t}}{\Gamma, x : A, \Gamma' \models^{2t} x : A} \; \text{(t-var)}$$

$$\frac{\Gamma, x : A \models^{2t} u : B : \mathsf{Prop}}{\Gamma \models^{2t} \lambda x^A . u : \forall x^A . B} \; \text{(t-abs)}$$

$$\frac{\Gamma \models^{2t} u : \forall x^A . B \quad \Gamma \models^{2t} v : A}{\Gamma \models^{2t} u \; v : B[x \leftarrow v]} \; \text{(t-app)}$$

$$\frac{\Gamma, x : A \models^{2t} B : \mathsf{Prop} \quad \sigma \, \mathsf{mot}_\Gamma \, \forall x^A . B}{\Gamma \models^{2t} \forall x^A . B : \mathsf{Prop}} \; \text{(t-prod)}$$

where $\sigma \, \mathsf{mot}_\Gamma \, C$ abbreviates :
- (a) σ exemplifies Γ, i.e. $\models^{2t} \sigma \cdot \Gamma$
- (b) and there is term t such that $\models^{2t} t : \sigma(C)$

A formal definition and some examples

With implicit partial exemplifications – λ_p^2

$$\frac{}{\emptyset \; \mathsf{wf}^{2p}} \text{ (p-env}_1)$$

$$\frac{\Gamma \vDash^{2p} A : \kappa \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : A \, \mathsf{wf}^{2p}} \text{ (p-env}_2)$$

$$\frac{\Gamma \, \mathsf{wf}^{2p}}{\Gamma \vDash^{2p} \quad \mathrm{Prop} : \mathrm{Type}} \text{ (p-ax)}$$

$$\frac{\Gamma, x : A, \Gamma' \, \mathsf{wf}^{2p}}{\Gamma, x : A, \Gamma' \vDash^{2p} x : A} \text{ (p-var)}$$

$$\frac{\Gamma, x : A \vDash^{2p} u : B : \mathrm{Prop}}{\Gamma \vDash^{2p} \lambda x^A.u : \forall x^A.B} \text{ (p-abs)}$$

$$\frac{\Gamma \vDash^{2p} u : \forall x^A.B \quad \Gamma \vDash^{2p} v : A}{\Gamma \vDash^{2p} u \; v : B[x \leftarrow v]} \text{ (p-app)}$$

$$\frac{\Gamma, x : A \vDash^{2p} B : \mathrm{Prop} \quad \sigma \; \widetilde{\mathrm{mot}}_\Gamma \; \forall x^A.B}{\Gamma \vDash^{2p} \forall x^A.B : \mathrm{Prop}} \text{ (p-prod)}$$

where $\sigma \; \widetilde{\mathrm{mot}}_\Gamma \; C$ abbreviates :

(a) σ partially exemplifies Γ, i.e. $\mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\Gamma)$

(b) and there is a term t such that $\sigma(\Gamma) \vDash^{2p} t : \sigma(C)$

A formal definition and some examples

With implicit partial exemplifications – λ_p^2

Theorems

- λ_p^2 equivalent to $\lambda_t^2 : \Gamma \vdash^{2p} t : A \quad \Leftrightarrow \quad \Gamma \vdash^{2t} t : A$

  $\triangleright$ exemplifications can be completed

- **λ_p^2 is a pedagogical sub-system of CC**

  $\triangleright$ definition exemplified !

General theorems

- λ_e^2, λ_t^2 and λ_p^2 are equivalents
- embeddings to and from pedagogical system F
- embedding from pedagogical λ^2

A formal definition and some examples
With implicit partial exemplifications – λ_p^2

But
Type checking not decidable in λ_e^2, λ_t^2 nor λ_p^2 :

$$\Gamma \models_\sigma^{2e} u : A \ ? \text{ dec.} \quad \Rightarrow \quad \Gamma' \models^{2t} v : B \ ? \qquad \text{dec.}$$
$$\Rightarrow \quad \exists? w \ \Delta \models^{fp} w : C \quad \text{dec.}$$
$$\Rightarrow \quad \exists? w' \ \Delta' \models^f w' : D \quad \text{dec.}$$

$$\rhd \exists? w' \ \Delta' \models^f w' : D \text{ undecidable } \text{[Urzyczyn (1997)]}$$

Idea
Annotate types with terms to ensure exemplification :

$$\frac{\Gamma_\sigma \vdash A_a : \kappa \quad x \notin \text{dom}(\Gamma)}{\Gamma_\sigma, x : A_a \text{ wf}} \ (\text{env}_2) \qquad \frac{\Gamma_\sigma, x : A_a \vdash B_b : \text{Prop} \quad \vdash t : \sigma(\forall x^{A_a}.B_b)}{\Gamma_\sigma \vdash (\forall x^{A_a}.B_b)_t : \text{Prop}} \ (\text{prod})$$

where $\Gamma_\sigma \equiv x_1 : A_{1_{a_1}}, \ldots, x_n : A_{n_{a_n}}$

Toward a Pedagogical Calculus of Constructions

Higher order

Goal

Obtain a Pedagogical λ^ω

New objects

- predicates/propositional functions
  - $\triangleright \lambda A^{\mathsf{Prop}}.\lambda B^{\mathsf{Prop}\to\mathsf{Prop}}.B\ A$
- postulated into environments
  - $\triangleright f : \mathsf{Prop} \to (\mathsf{Prop} \to \mathsf{Prop}) \to \mathsf{Prop}$
- $\Rightarrow$ notion of predicate exemplification needed

Idea

Exemplifiable predicate $\Leftrightarrow$ useful propositional function

Exemplifiable predicate (formally)

$P : \mathcal{O}_1 \to \ldots \to \mathcal{O}_n \to$ Prop exemplifiable if :

- can be completely applied $\qquad \triangleright$ there are $u_i : \mathcal{O}_i$
- reducible to an exemplifiable type $\qquad \triangleright P\ \vec{u} \overset{*}{\rightsquigarrow} R$ and $\vdash \sigma \cdot R$

Abbreviated : $\sigma\, \mathrm{mot}^{\mathcal{O}_1 \to \ldots \to \mathcal{O}_n \to \mathrm{Prop}}(P)$

Examplifiable predicates

$P := \lambda A^{\mathsf{Prop}}.A \to A \qquad\qquad\qquad \triangleright P\ \top \overset{*}{\rightsquigarrow} \top \to \top$

$Q := \lambda A^{\mathsf{Prop}}.\lambda B^{\mathsf{Prop} \to \mathsf{Prop}}.B\ A \qquad \triangleright Q\ \top\ (\lambda C^{\mathsf{Prop}}.\top) \overset{*}{\rightsquigarrow} \top$

Non exemplifiable predicate

$S := \lambda A^{\mathsf{Prop}}.\forall B^{\mathsf{Prop}}.A \leftrightarrow B$

$V := \lambda A^{\mathsf{Prop}}.\forall B^{\mathsf{Prop}}.A \to B \qquad\qquad \triangleright$ hopefully no arguments

Beware interaction of exemplifiable predicates

- $Id := \lambda A^{\mathsf{Prop}}.A$ and $Q := \lambda R^{\mathsf{Prop}\to\mathsf{Prop}}.\forall F^{\mathsf{Prop}}.R\ F$

$$\triangleright \text{ but } Q\ Id \overset{*}{\leadsto} \bot$$

Necessary constraint

$$\frac{\Gamma \Vvdash_{\sigma} u : A \to B : \mathsf{Type} \quad \Gamma \Vvdash_{\sigma} v : A \quad \sigma\,\mathsf{mot}^{B}(u\ v)}{\Gamma \Vvdash_{\sigma} u\ v : B} \ (\omega e\text{-app}_{\square})$$

Toward a Pedagogical Calculus of Constructions

Pedagogical higher-order λ-calculus $- \lambda_e^\omega$

Beware interaction of exemplifiable predicates

- $Id := \lambda A^{\mathsf{Prop}}.A$ and $Q := \lambda R^{\mathsf{Prop}\to\mathsf{Prop}}.\forall F^{\mathsf{Prop}}.R\ F$

  $\triangleright$ but $Q\ Id \overset{*}{\rightsquigarrow} \bot$

- $S := \lambda A^{\mathsf{Prop}\to\mathsf{Prop}}.\lambda H^{\forall B^{\mathsf{Prop}}.A\ B}.H$ $\triangleright$ but $S\ Id \overset{*}{\rightsquigarrow} \lambda H^{\bot}.H$

Necessary constraint

$$\frac{\Gamma \vDash_{\sigma} u : \forall x^A.B : \mathsf{Prop} \quad \Gamma \vDash_{\sigma} v : A \quad \sigma\,\mathsf{mot}^{\mathsf{Prop}}(B[x \leftarrow v])}{\Gamma \vDash_{\sigma} u\ v : B[x \leftarrow v]}\ (\omega e\text{-app}_{\star})$$

Toward a Pedagogical Calculus of Constructions

Pedagogical higher-order λ-calculus – λ_e^ω

Beware interaction of exemplifiable predicates

- $Id := \lambda A^{\mathsf{Prop}}.A$ and $Q := \lambda R^{\mathsf{Prop} \to \mathsf{Prop}}.\forall F^{\mathsf{Prop}}.R \; F$

  $\triangleright$ but $Q \; Id \overset{*}{\leadsto} \bot$

- $S := \lambda A^{\mathsf{Prop} \to \mathsf{Prop}}.\lambda H^{\forall B^{\mathsf{Prop}}.A \; B}.H$

  $\triangleright$ but $S \; Id \overset{*}{\leadsto} \lambda H^{\bot}.H$

Necessary constraint

$$\frac{\Gamma \vdash_{\overline{\sigma}} u : \forall x^A.B : \mathsf{Prop} \quad \Gamma \vdash_{\overline{\sigma}} v : A \quad \textcolor{red}{\sigma \, \mathsf{mot}^{\mathsf{Prop}}(B[x \leftarrow v])}}{\Gamma \vdash_{\overline{\sigma}} u \; v : B[x \leftarrow v]} \; (\omega e\text{-app}_\star)$$

Strong : only normal form examples ?

Toward a Pedagogical Calculus of Constructions

Pedagogical higher-order λ-calculus – λ_e^ω

Theorems

- λ_e^ω satisfies Poincaré criterion
- λ_e^ω satisfies the converse Poincaré criterion
- subject reduction still conjectured

  ▷ usual substitution lemma invalid :

  $A : \star \to \star \vdash_{\lambda_z \star . \top} \lambda H^{\forall B^\star . A\ B} . H : (\forall B^\star . A\ B) \to (\forall B^\star . A\ B) : \star$
  $\vdash_\emptyset \lambda C^\star . C : \star \to \star$

  $\not\vdash_\emptyset \lambda H^{\forall B^\star . (\lambda C^\star . C)\ B} . H : (\forall B^\star . (\lambda C^\star . C)\ B) \to (\forall B^\star . (\lambda C^\star . C)\ B)$

  where $\star := \mathsf{Prop}$

Toward a Pedagogical Calculus of Constructions
Summary of necessary constraints

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \forall x^A.B : s_2} \text{ (prod)}$$

(s_1, s_2)	λ^2	λ^ω
(Prop, Prop)	✓	✓
(Type, Prop)	✗	✗
(Prop, Type)		
(Type, Type)		✓

- ✓ instance of the rule does not produce empty type
- ✗ instance of the rule can produce empty types

Toward a Pedagogical Calculus of Constructions

Summary of necessary constraints

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \forall x^A.B : s_2} \; (\text{prod})$$

(s_1, s_2)	λ^2	λ^ω	λC
(Prop, Prop)	✓	✓	✗
(Type, Prop)	✗	✗	✗
(Prop, Type)			✓
(Type, Type)		✓	✓

- ► ✓ instance of the rule does not produce empty type
- ► ✗ instance of the rule can produce empty types

Toward a Pedagogical Calculus of Constructions
Summary of necessary constraints

$$\frac{\Gamma \vdash v : A : s_1 \quad \Gamma \vdash u : \forall x^A.B : s_2}{\Gamma \vdash u \; v : B[x \leftarrow v]} \; \text{(app)}$$

(s_1, s_2)	λ^2	λ^ω
(Prop, Prop)	✓	✓
(Type, Prop)	✓	✗
(Prop, Type)		
(Type, Type)		✗

- ▶ ✓ instance of the rule does not produce empty type
- ▶ ✗ instance of the rule can produce empty types

Toward a Pedagogical Calculus of Constructions

Summary of necessary constraints

$$\frac{\Gamma \vdash v : A : s_1 \quad \Gamma \vdash u : \forall x^A.B : s_2}{\Gamma \vdash u\ v : B[x \leftarrow v]} \text{ (app)}$$

(s_1, s_2)	λ^2	λ^ω	λC
(Prop, Prop)	✓	✓	✗
(Type, Prop)	✓	✗	✗
(Prop, Type)			✗
(Type, Type)		✗	✗

- ► ✓ instance of the rule does not produce empty type
- ► ✗ instance of the rule can produce empty types

Toward a Pedagogical Calculus of Constructions

Pedagogical Calculus of Constructions ?

$$\frac{}{\emptyset \, \mathsf{wf}^{c_e}_{\sigma}} \, (\mathsf{env}_1)$$

$$\frac{\Gamma \vDash^{c_e}_{\sigma} A : \kappa \quad \vDash^{c_e}_{\emptyset} a : A' \quad \sigma(A) \overset{*}{\rightsquigarrow} A' \quad x \notin \mathsf{dom}(\Gamma)}{\Gamma, x : A \, \mathsf{wf}^{c_e}_{\sigma::(x \,\mapsto\, a)}} \, (\mathsf{env}_2)$$

$$\frac{\Gamma \, \mathsf{wf}^{c_e}_{\sigma}}{\Gamma \vDash^{c_e}_{\sigma} o : \top : \mathsf{Prop} : \mathsf{Type}} \, (\mathsf{ax})$$

$$\frac{\Gamma, x : A, \Gamma' \, \mathsf{wf}^{c_e}_{\sigma}}{\Gamma, x : A, \Gamma' \vDash^{c_e}_{\sigma} x : A} \, (\mathsf{var})$$

$$\frac{\Gamma, x : A \vDash^{c_e}_{\sigma::(x \,\mapsto\, a)} u : B : \kappa}{\Gamma \vDash^{c_e}_{\sigma} \lambda x^A.u : \forall x^A.B} \, (\mathsf{abs})$$

Toward a Pedagogical Calculus of Constructions
Pedagogical Calculus of Constructions ?

$$\frac{\Gamma \vDash_{\sigma}^{\mathcal{C}_e} u : \forall x^A.B : \mathsf{Prop} \quad \Gamma \vDash_{\sigma}^{\mathcal{C}_e} v : A \quad \sigma\,\mathsf{mot}^{\mathsf{Prop}}(B[x \leftarrow v])}{\Gamma \vDash_{\sigma}^{\mathcal{C}_e} u\ v : B[x \leftarrow v]}\ (\mathsf{app}_\star)$$

$$\frac{\Gamma \vDash_{\sigma}^{\mathcal{C}_e} u : \forall x^A.B : \mathsf{Type} \quad \Gamma \vDash_{\sigma}^{\mathcal{C}_e} v : A \quad \sigma\,\mathsf{mot}^{B[x \leftarrow v]}(u\ v)}{\Gamma \vDash_{\sigma}^{\mathcal{C}_e} u\ v : B[x \leftarrow v]}\ (\mathsf{app}_\square)$$

$$\frac{\Gamma, x : A \vDash_{\sigma::(x \mapsto a)}^{\mathcal{C}_e} B : \mathsf{Prop} \quad \sigma\,\mathsf{mot}^{\mathsf{Prop}}(\forall x^A.B)}{\Gamma \vDash_{\sigma}^{\mathcal{C}_e} \forall x^A.B : \mathsf{Prop}}\ (\mathsf{prod}_\star)$$

$$\frac{\Gamma, x : A \vDash_{\sigma::(x \mapsto a)}^{\mathcal{C}_e} B : \mathsf{Type}}{\Gamma \vDash_{\sigma}^{\mathcal{C}_e} \forall x^A.B : \mathsf{Type}}\ (\mathsf{prod}_\square)$$

$$\frac{\Gamma \vDash_{\sigma}^{\mathcal{C}_e} u : A \quad A \simeq A' \quad \Gamma \vDash_{\sigma}^{\mathcal{C}_e} A' : \kappa}{\Gamma \vDash_{\sigma}^{\mathcal{C}_e} u : A'}\ (\mathsf{conv})$$

Conclusion and further work

Contributions

- formal definition of pedagogical sub-system of CC
- examples of this definition (λ_e^2, λ_t^2, λ_p^2, λ_e^ω)
- introduction of explicit examplifications
- study of type checking pedagogical calculi
- motivated conjectures : pedagogical higher-order and Calculus of Constructions

Further work

- show the conjectures
- study machine implementation
- extend to stronger systems
- formal verification of Griss' negationless mathematics
- inconsistency and pedagogy...